

On hyperballeans of bounded geometry

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Abstract. A ballean (or coarse structure) is a set endowed with some family of subsets, the balls, in such a way that ballians with corresponding morphisms can be considered as asymptotic counterparts of uniform topological spaces. For a ballean \mathcal{B} on a set X , the hyperballean \mathcal{B}^b is a ballean naturally defined on the set X^b of all bounded subsets of X . We describe all ballians with hyperballians of bounded geometry and analyze the structure of these hyperballians.

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1 Introduction and preliminaries

Following [7], [8], we say that a *ball structure* is a triple $\mathcal{B} = (X, P, B)$, where X, P are non-empty sets, and for all $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called a *ball of radius* α around x . It is supposed that $x \in B(x, \alpha)$ for all $x \in X$, $\alpha \in P$. The set X is called the *support* of \mathcal{B} . P is called the *set of radii*.

Given any $x \in X$, $A \subseteq X$, $\alpha \in P$, we set

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\}, \quad B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha).$$

A ball structure $\mathcal{B} = (X, P, B)$ is called a *ballean* if

- for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B(x, \alpha) \subseteq B^*(x, \alpha'), \quad B^*(x, \beta) \subseteq B(x, \beta');$$

- for every $\alpha, \beta \in P$, there exist $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma);$$

- for any $x, y \in X$, there exists $\alpha \in P$ such that $y \in B(x, \alpha)$.

We note that a ballean can be considered as an asymptotic counterpart of a uniform space, and could be defined [9] in terms of entourages of the diagonal Δ_X in $X \times X$. In this case a ballean is called a *coarse structure*. For categorical look at the ballean and coarse structures as "two faces of the same coin" see [4].

Let $\mathcal{B} = (X, P, B)$, $\mathcal{B}' = (X', P', B')$ be ballians. A mapping $f : X \rightarrow X'$ is called *coarse* if, for every $\alpha \in P$, there exists $\alpha' \in P'$ such that

$$f(B(x, \alpha)) \subseteq B'(f(x), \alpha').$$

A bijection $f : X \rightarrow X'$ is called an *asymorphism* between \mathcal{B} and \mathcal{B}' if f and f^{-1} are coarse mappings. In this case \mathcal{B} and \mathcal{B}' are called *asymorphic*. If $X = X'$ and the identity mapping $id : X \rightarrow X'$ is an asymorphism, we identify \mathcal{B} and \mathcal{B}' , and write $\mathcal{B} = \mathcal{B}'$. Given any ballean $\mathcal{B} = (X, P, B)$, replacing each ball $B(x, \alpha)$ to $B(x, \alpha) \cap B^*(x, \alpha)$, we get the same ballean, so in what follows we suppose that $B(x, \alpha) = B^*(x, \alpha)$.

Let $\mathcal{B} = (X, P, B)$ be a ballean. Each non-empty subset Y of X defines a *subballean* $\mathcal{B}_Y = (Y, P_Y, B_Y)$, where $B_Y(y, \alpha) = Y \cap B(y, \alpha)$. A subset Y is called *large* if $X = B(Y, \alpha)$ for some $\alpha \in P$. Two ballians \mathcal{B} and \mathcal{B}' with the support X and X' are called *coarsely equivalent* if there exist large subsets $Y \subseteq X$ and $Y' \subseteq X'$ such that the ballians \mathcal{B}_Y and $\mathcal{B}'_{Y'}$ are asymorphic.

For a ballean $\mathcal{B} = (X, P, B)$, a subset Y of X is called *bounded* if there exist $x \in X$ and $\alpha \in P$ such that $Y \subseteq B(x, \alpha)$. A ballean \mathcal{B} is called *bounded* if the support X is bounded. Each bounded ballean is coarsely equivalent to a ballean whose support is a singleton.

Now we are ready to introduce the main subject of the note. For a ballean $\mathcal{B} = (X, P, B)$, we denote by X^b the family of all non-empty bounded subsets of, consider the ballean $\mathcal{B}^b = (X^b, P, B^b)$, where

$$B^b(Y, \alpha) = \{Z \in X^b : Z \subseteq B(Y, \alpha), Y \subseteq B(Z, \alpha)\},$$

and say that \mathcal{B}^b is the *hyperballean* of \mathcal{B} .

For $\alpha \in P$, a subset S of X is called α -discrete if $B(x, \alpha) \cap S = \{x\}$ for each $x \in S$. We say that \mathcal{B} is of *bounded geometry* if there exist $\alpha \in P$ and a function $f : P \rightarrow \mathbb{N}$ such that if S is an α -discrete subset of a ball $B(x, \beta)$ then $|S| \leq f(\beta)$. A ballean \mathcal{B} is called *uniformly locally finite* if, for every $\beta \in P$, there is $n(\beta) \in \mathbb{N}$ such that $|B(x, \beta)| \leq n(\beta)$ for every $x \in X$. By [6], \mathcal{B} is of bounded geometry if and only if there exists large subset Y of X such that \mathcal{B}_Y is uniformly locally finite.

It should be mentioned that the notion of bounded geometry went from asymptotic topology where metric spaces of bounded geometry

play the central part [5]. For interrelations between balleanes of bounded geometry and G -spaces see [6].

Every metric space (X, d) defines the *metric ballean* (X, \mathbb{R}^+, B_d) , where $B_d(x, r) = \{y \in X : d(x, y) \leq r\}$. A ballean \mathcal{B} is called *metrizable* if \mathcal{B} is asymorphic to some metric ballean. By [8, Theorem 2.1.1], for a ballean \mathcal{B} , the following statements are equivalent: \mathcal{B} is metrizable, \mathcal{B} is coarsely equivalent to some metrizable ballean, the set P has a countable confinal subset S . We recall S is confinal if, for every $\beta \in P$ there is $\alpha \in S$ such that $\alpha > \beta$. Here $\alpha > \beta$ means that $B(x, \beta) \subseteq B(x, \alpha)$ for each $x \in X$. Applying this criterion, we conclude that, for every metrizable ballean \mathcal{B} , the hyperballean \mathcal{B}^b is metrizable.

2 Results

For a non-empty set X and the family \mathcal{F}_X of all finite subsets of X , we denote by \mathbf{F}_X the ballean $(X, \mathcal{F}_X, B_{\mathbf{F}})$ where

$$B_{\mathbf{F}}(x, F) = \begin{cases} \{x\} & \text{if } x \notin F; \\ F & \text{if } x \in F. \end{cases}$$

Then $\mathbf{F}_X^b = \{\mathfrak{F}_X \setminus \{\emptyset\}, \mathfrak{F}_X, B_{\mathbf{F}}^b\}$, where $B_{\mathbf{F}}^b(H, F) = \{H\}$ if $H \cap F = \emptyset$ and $B_{\mathbf{F}}(H, F) = \{(H \setminus F) \cup Z : Z \subseteq H, Z \neq \emptyset\}$ otherwise.

The ballean $\mathbf{F}_{\omega, \omega} = \{0, 1, \dots\}$ is metrizable (say, by the metric $d(m, n) = |2^m - 2^n|$), so \mathbf{F}_{ω}^b is also metrizable (say, by the Hausdorff metric \mathfrak{b}_H). At the end of the note, we point out some more explicit metrization of \mathbf{F}_{ω}^b .

Theorem 2.1. *For an unbounded ballean $\mathcal{B} = (X, P, B)$, the following statements hold:*

- (i) \mathcal{B}^b is uniformly locally finite if and only if $\mathcal{B} = \mathbf{F}_X$;
- (ii) \mathcal{B}^b is of bounded geometry if and only if there exists a large subset Y of X such that $\mathcal{B}_Y = \mathbf{F}_Y$.

For a cardinal κ , we denote by \mathbf{Q}_{κ} the ballean with the support

$$\mathbf{Q}_{\kappa} = \{(x_{\alpha})_{\alpha < \kappa} : x_{\alpha} \in \{0, 1\}, x_{\alpha} = 0$$

$$\text{for all but finitely many } \alpha < \kappa\},$$

the set of radii \mathcal{F}_{κ} and the balls

$$B_{\mathbf{Q}}((x_{\alpha})_{\alpha < \kappa}, F) = \{(y_{\alpha})_{\alpha < \kappa} : x_{\alpha} = y_{\alpha} \text{ for all } \alpha \in \kappa \setminus F\}.$$

The ballean \mathbf{Q}_ω is known as the Cantor macrocube and sometimes is denoted by $2^{<\omega}$ or $2^{<\mathbb{N}}$. For characterization of balleanes coarsely equivalent to the Cantor macrocube see [3]. In [1], $2^{<\kappa}$ denotes the ballean of all $\{0, 1\}$ κ -sequences $(x_\alpha)_{\alpha < \kappa}$ such that $|\{\alpha < \kappa : x_\alpha = 1\}| < \kappa$.

A ballean $\mathcal{B} = (X, P, B)$ is called *asymptotically scattered* if, for every unbounded subset Y of X , there is $\alpha \in P$, such that, for every $\beta \in P$, there exists $y \in Y$ such that

$$(B(y, \beta) \setminus B(y, \alpha)) \bigcap Y = \emptyset.$$

For asymptotically scattered subballeanes of group balleanes see [2].

For a ballean $\mathcal{B} = (X, P, B)$, the subset Y, Z of X are called *close* if there exists $\alpha \in P$ such that $Y \subseteq B(Z, \alpha)$, $Z \subseteq B(Y, \alpha)$.

Theorem 2.2. *Let κ be an infinite cardinal, $n \in \mathbb{N}$, $[\kappa]^n = \{F \subset \kappa : |F| = n\}$, $x \in \kappa$. Then the following statements hold:*

- (i) *the subballean of \mathbf{F}_κ^b with the support $[\kappa]^n$ is asymptotically scattered;*
- (ii) *the subballean of \mathbf{F}_κ^b with the support $\{F \in \mathfrak{F}_\kappa : x \in F\}$ is asyomorphic to \mathbf{Q}_κ ;*
- (iii) *\mathbf{F}_ω^b can be partitioned into countably many pairwise close Cantor macrocubes but \mathbf{F}_ω^b is not coarsely equivalent to \mathbf{Q}_ω .*

At the end of the note, we describe some explicit asyomorphic embedding of \mathbf{F}_ω^b into \mathbf{Q}_ω .

3 Proofs

Proof of Theorem 1.2. (i) By the definition of balls in \mathbf{F}_ω^b , \mathbf{F}_ω^b is uniformly locally finite.

If the identity mapping $id : X \longrightarrow X$ is not an asyomorphism between \mathcal{B} and \mathbf{F}_X then we can choose $\alpha \in P$ and a sequence $(x_n)_{n < \omega}$ in X such that $|B(x_n, \alpha)| > 1$ and $B(x_i, \alpha) \cap B(x_j, \alpha) = \emptyset$ for all $i < j < \omega$. For each $i < \omega$, we pick $y_i \in B(x_i, \alpha)$, $y_i \neq x_i$, put $X_n = \{x_0, \dots, x_n\}$, $X_{n,i} = X_n \cup \{y_i\}$, $i \leq n < \omega$. Then $X_{n,i} \in B^b(X_n, \alpha)$, so $|B^b(X_n, \alpha)| > n$ and \mathcal{B} is not uniformly locally finite.

(ii) We assume that Y is a large subset of X and choose $\beta \in P$ such that $B(Y, \beta) = X$. For each $x \in X$, we pick $y_x \in Y$ such that $y_x \in B(x, \beta)$. If $F \in X^b$ then $\{y_x : x \in F\} \in Y^b$ and $F \in B^b(\{y_\kappa : x \in F\}, \beta)$. It follows that $B^b(Y^b, \beta) = X^b$, Y^b is large in X^b so \mathcal{B}_Y^b and \mathcal{B}^b are coarsely equivalent. In particular, if $\mathcal{B}_Y = \mathbf{F}_Y$, we conclude that \mathcal{B} is of bounded geometry.

We suppose that \mathcal{B}^b is of bounded geometry and let $\alpha \in P$ and $f : P \rightarrow \mathbb{N}$ witness this property. Using Zorn's lemma, we choose a maximal by inclusion subset Y of X such that $B(y, \alpha) \cap B(y', \alpha) = \emptyset$ for all distinct $y, y' \in Y$. We show that $\mathcal{B}_Y = \mathbf{F}_Y$.

If the identity mapping $id : Y \rightarrow Y$ is not an asyrmorphism between \mathcal{B}_Y and \mathbf{F}_Y then there are $\beta \in P$ and a sequence $(y_n)_{n \in \omega}$ in Y such that $|B_Y(y_n, \beta)| > 1$ and $B_Y(y_i, \beta) \cap B_Y(y_j, \beta) = \emptyset$ for all $i < j < \omega$. For each $i < \omega$, we pick $z_i \in B_Y(y_i, \beta)$, $z_i = y_i$, put $Y_n = \{y_0, \dots, y_n\}$, $Y_{n,i} = Y_n \cup \{y_i\}$, $i \leq n < \omega$. Then $Y_{n,i} \in B^b(Y_n, \beta)$ and the set $\{Y_{n,i} : i \leq n\}$ is α -discrete. Thus, for $n > f(\beta)$ we get a contradiction with the choice of α and f . \square

Proof of Theorem 2.2. (i) We say that a subset of a ballean is *asymptotically scattered* if corresponding subballean has this property. We use the following observation: the union of two asymptotically scattered subsets is asymptotically scattered (see [2]).

We note that every unbounded subset in \mathbf{F}_κ^b is infinite and proceed on induction by n . For $n = 1$, the statement is evident: given any $H \in \mathfrak{F}_\kappa$ and an infinite subset Y of $[\kappa]^1$, we take $\{y\} \in Y$, $y \in H$ and get $B_{\mathbf{F}}(\{y\}, H) = \{y\}$.

Assuming that the statement is true for $[\kappa]^n$, let Y be an infinite subset of $[\kappa]^{n+1}$. For each $F \in Y$, we denote by $\min F$ and $\max F$, the minimal and maximal elements of F with respect to the ordinal ordering of κ and consider two cases.

Case: the set $\{\min F : F \in Y\}$ is infinite. We take an arbitrary $H \in \mathfrak{F}_\kappa$ and choose $F \in Y$ such that $\max H < \min F$. Then $B_{\mathbf{F}}^b(F, H) = \{F\}$.

Case: the set $\{\min F : F \in Y\}$ is finite, $\{\min F : F \in Y\} = x_1, \dots, x_n$. For each $i \in \{1, \dots, n\}$, we denote $Z_i = \{F \in [\kappa]^{n+1} : x_i \in F\}$. We note that Z_i is asyrmorphic to $[\kappa]^n$ and, by the inductive assumption, Z_i is asymptotically scattered.

Then $Z_1 \cup \dots \cup Z_n$ is asymptotically scattered, $Y \subseteq Z_1 \cup \dots \cup Z_n$ and we can use definition of asymptotically scattered subsets to choose $\alpha \in \mathfrak{F}_X$ suitable for Y .

(ii) We use the standard bijection $\chi : \mathfrak{F}_\kappa \rightarrow Q_\kappa$ defined by $\chi(K) = (x_\alpha)_{\alpha < \kappa}$, where $x_\alpha = 1$ if and only if $\alpha \in K$. Then the restriction of χ to $\{F \in \mathfrak{F}_\kappa : x \in F\}$ is a asyrmorphic embedding. Indeed, to verity this property we may use as radii in \mathbf{F}^b only balls containing x . Clearly, $\chi\{F \in \mathfrak{F}_\kappa : \kappa \in F\}$ is asyrmorphic to \mathbf{Q}_κ . \square

(iii) For every $n \in \omega$, let $\mathcal{M}_n = \{F \in \mathfrak{F}_X : \min F = n\}$. Applying (ii), we see that \mathcal{M}_n is asyrmorphic to \mathbf{Q}_ω . We take arbitrary $i, j \in \omega$, denote $m = \max\{i, j\}$, $I_m = \{0, \dots, m\}$. Then $\mathcal{M}_i \subseteq B_{\mathbf{F}}^b(\mathcal{M}_j, I_m)$, $\mathcal{M}_j \subseteq B_{\mathbf{F}}^b(\mathcal{M}_i, I_m)$ so $\mathcal{M}_i, \mathcal{M}_j$ are close.

Given $H \in \mathfrak{F}_\omega$, we take $F \in \mathfrak{F}_\omega$ such that $\max H < \min F$. Then

$B_{\mathbf{F}}^b(F, H) = \{F\}$. In terminology of [3], it means that \mathbf{F}_{ω}^b has an asymptotically isolated balls but every ballean coarsely equivalent to \mathbf{Q}_{ω} has no isolated balls. \square

To embed asymptotically \mathbf{F}_{ω}^b into \mathbf{Q}_{ω} , we use $2\mathbb{N}$ in place of ω . We define a mapping $f : \mathfrak{F}_{2\mathbb{N}} \setminus \{\emptyset\} \rightarrow Q_{\omega}$ by the $f(K) = (x_n)_{n < \omega}$, where $x_n = 1$ if and only if $n \in \{\min K - 1\} \cup K$. We note that the set $S = f(\mathfrak{F}_{2\mathbb{N}} \setminus \{\emptyset\})$ consists of all sequences $(x_n)_{n < \omega}$ with at least two non-zero coordinates and such that the first non-zero coordinate of $(x_n)_{n < \omega}$ is odd and all other are even. For each $K \in \mathfrak{F}_{2\mathbb{N}} \setminus \{\emptyset\}$ and $n \in \mathbb{N}$, we have

$$f(B_{\mathbf{F}}(K, \{2, 4, \dots, 2n\})) = S \cap B_{\mathbf{Q}}(f(K), \{1, 2, \dots, 2n\}),$$

witnessing that f is an asyomorphic embedding of $\mathbf{F}_{2\mathbb{N}}^b$ into \mathbf{Q}_{ω} .

With this representation, \mathbf{F}_{ω}^b can be easily metrizable by means of restriction to S of the stadard metric d on $Q_{\omega} : d((x_n)_{n \in \omega}, (y_n)_{n \in \omega}) = \min\{m : x_n = y_n \text{ for all } n \geq m\}$.

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